Derivation of the state matrix for dynamic analysis of linear homogeneous media

Juan Pablo Parra Martinez,[1,a)] Olivier Dazel,[2] Peter Göransson,[1,a)] and Jacques Cuenca[3,b)]

[1]*MWL/Centre for ECO[2] Vehicle Design, KTH Royal Institute of Technology, Teknikringen 8, SE-10044 Stockholm, Sweden*

[2]*Laboratoire d'Acoustique de l'Université du Maine–UMR CNRS 6613, Avenue Olivier Messiaen, F-72085 Le Mans Cedex, France*

[3]*Siemens Industry Software, Interleuvenlaan 68, B-3001 Leuven, Belgium*

jppm@kth.se, olivier.dazel@univ-lemans.fr, pege@kth.se, jacques.cuenca@siemens.com

Abstract: A method to obtain the state matrix of an arbitrary linear homogeneous medium excited by a plane wave is proposed. The approach is based on projections on the eigenspace of the governing equations matrix. It is an alternative to manually obtaining a linearly independent set of equations by combining the governing equations. The resulting matrix has been validated against previously published derivations for an anisotropic poroelastic medium.

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1. State form of the governing equations

The dynamic behaviour of any homogeneous linear physical medium, under harmonic excitation ω, can be modelled by a general system of first-order equations in a Cartesian coordinate system xyz,

$$\left(\mathbf{M}+\mathbf{A}_x\frac{\partial}{\partial x}+\mathbf{A}_y\frac{\partial}{\partial y}+\mathbf{A}_z\frac{\partial}{\partial z}\right)\mathbf{w}(x,y,z)\,e^{\imath\omega t}=\mathbf{0}, \tag{1}$$

where $\mathbf{w}$ is a vector of physical field variables used in the modelling. The size of the differential system is denoted by m which depends on the medium and the chosen formulation to represent its physics. For a homogeneous medium the matrices $\mathbf{M}$, $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{A}_z$ in Eq. (1) are constant, frequency dependent and, in general, complex (to take dissipation into account). Their expressions are straightforward to derive from first principles, i.e., from the involved conservation relations.

It is assumed that the medium is a constituent of a multilayer system with interfaces normal to the z direction and that this system is excited by a plane wave with prescribed wavenumbers k_x, k_y. Partial differentiation with respect to coordinates x and y become algebraic operations such that Eq. (1) may be rewritten as

$$\left(\mathbf{R}+\mathbf{A}_z\frac{\partial}{\partial z}\right)\mathbf{w}(z)\,e^{\imath\left(\omega t-k_x x-k_y y\right)}=\mathbf{0}, \tag{2}$$

where

$$\mathbf{R}=\mathbf{M}-\imath k_x\mathbf{A}_x-\imath k_y\mathbf{A}_y. \tag{3}$$

Hence, $\mathbf{w}$ depends on z and only the partial derivation with respect to z remains. The rank of the system in Eq. (2) is n, with $n<m$, as a consequence of the field variables in $\mathbf{w}(z)$ and the plane wave excitation.

In the following, the common spatial and time dependence $e^{\imath(\omega t-k_x x-k_y y)}$ is omitted.

Out of the m field amplitudes in $\mathbf{w}(z)$, only n is linearly independent. This linear dependence among the fields in $\mathbf{w}(z)$ is partially introduced by the spatial dependence prescribed by the wavenumbers k_x, k_y. To account for this, a partitioning of $\mathbf{w}(z)$ is *a priori* introduced, as

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[a)]Also at: Laboratoire d'Acoustique de l'Universite du Maine–UMR CNRS 6613, Avenue Olivier Messiaen, F-72085 Le Mans Cedex, France.

[b)]Also at: MWL/Centre for ECO2 Vehicle Design, KTH Royal Institute of Technology, Teknikringen 8, SE-10044 Stockholm, Sweden.

$$\mathbf{w}(z) = \begin{Bmatrix} \mathbf{s}(z) \\ \mathbf{s}_0(z) \end{Bmatrix}. \tag{4}$$

s(z) is a state vector of the system of length n. The actual partitioning in the previous equation (i.e., the choice of the variables which form the state vector), is trivial for most media, as the components in $\mathbf{s}(z)$ are those required to establish the coupling relations at the interfaces of the medium. Examples of state vectors for different applications may be found in the literature.[1–3] In particular, Eqs. (19) and (20) in Brouard *et al.*[4] present the state vector for isotropic poroelastic media for two different material models. In the case of an anisotropic porous medium, Eqs. (21) and (22) in Parra Martinez *et al.*[5] present the state vector for the Biot model in the $\{\mathbf{u}^s, \mathbf{u}^f\}$ representation, illustrating the partitioning in Eq. (4).

The evolution of $\mathbf{s}(z)$ can be rewritten in the form of a state space representation,[6–8]

$$\frac{\partial}{\partial z}\mathbf{s}(z) = -\boldsymbol{\alpha}\,\mathbf{s}(z), \tag{5}$$

where $\boldsymbol{\alpha}$ is referred to as the state matrix. In a recent paper,[5] the authors proposed a method to model the acoustics of fully anisotropic poroelastic multilayered systems. A crucial step in the derivation was to determine the state matrix $\boldsymbol{\alpha}$, and was shown to involve a succession of tedious manipulations of the governing equations. Although that approach did not involve any intrinsic difficulties, such a manipulation is only possible on a case-by-case basis.

Here, an alternative, systematic method for determining $\boldsymbol{\alpha}$ is proposed. The key advantage is that it avoids manual rearrangements, using as inputs the $\mathbf{R}$ and $\mathbf{A}_z$ matrices in Eq. (2). Thus, it provides not only a way to check the previously proposed method but also its general form allows for any linear homogeneous physical system to be treated, and is hence not limited to porous materials *per se*.

2. Derivation of the state matrix

The generalised un-symmetric eigenvalue problem associated to $\mathbf{R}$ and $\mathbf{A}_z$ can be written as

$$\mathbf{A}_z = \mathbf{R}\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Psi},\ \boldsymbol{\Psi} = \boldsymbol{\Phi}^{-1}, \tag{6}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of m eigenvalues and $\boldsymbol{\Phi}$ is a matrix of eigenvectors of size $(m \times m)$, corresponding to Eq. (6). Due to the rank deficiency, $q = m - n$ eigenvalues in $\boldsymbol{\Lambda}$ are equal to zero. Thus, reordering the eigenvectors such that the first n columns in $\boldsymbol{\Phi}$ and the first n rows in $\boldsymbol{\Psi}$ correspond to the non-zero eigenvalues yields

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \boldsymbol{\Phi} = [\boldsymbol{\Phi}_e \boldsymbol{\Phi}_0],\ \boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_e \\ \boldsymbol{\Psi}_0 \end{bmatrix}, \tag{7}$$

where the dimension of $\boldsymbol{\Lambda}_e$ is $(n \times n)$, $\boldsymbol{\Phi}_e$ is $(m \times n)$, $\boldsymbol{\Psi}_e$ is $(n \times m)$, $\boldsymbol{\Phi}_0$ is $(m \times q)$, and $\boldsymbol{\Psi}_0$ is $(q \times m)$. Considering Eq. (7), Eq. (2) reads

$$\mathbf{R}\left(\mathbf{I}_m + \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Psi}\frac{\partial}{\partial z}\right)\mathbf{w}(z) = \mathbf{0}, \tag{8}$$

where $\mathbf{I}_m$ is the identity matrix of size m. Equation (8) may be simplified due to the zero eigenvalues

$$\mathbf{R}\left(\mathbf{I}_m + \boldsymbol{\Phi}_e\boldsymbol{\Lambda}_e\boldsymbol{\Psi}_e\frac{\partial}{\partial z}\right)\mathbf{w}(z) = 0. \tag{9}$$

This system can be split into

$$\mathbf{R}_e\left(\mathbf{I}_m + \Phi_e\Lambda_e\Psi_e\frac{\partial}{\partial z}\right)\mathbf{w}(z) = \mathbf{0}, \quad \mathbf{R}_0\left(\mathbf{I}_m + \Phi_e\Lambda_e\Psi_e\frac{\partial}{\partial z}\right)\mathbf{w}(z) = \mathbf{0} \tag{10}$$

with

$$\mathbf{R}_e = \boldsymbol{\Psi}_e\,\mathbf{R},\ \mathbf{R}_0 = \boldsymbol{\Psi}_0\,\mathbf{R}. \tag{11}$$

The first system in Eq. (10) is considered in order to cancel the partial derivative over z,

$$\boldsymbol{\Psi}_e\frac{\partial}{\partial z}\mathbf{w}(z) = -(\mathbf{R}_e\boldsymbol{\Phi}_e\boldsymbol{\Lambda}_e)^{-1}\mathbf{R}_e\,\mathbf{w}(z). \tag{12}$$

This expression can be introduced in the second system in Eq. (10),

$$\mathbf{R}_0(\mathbf{I}_m - \mathbf{\Phi}_e\mathbf{\Lambda}_e(\mathbf{R}_e\mathbf{\Phi}_e\mathbf{\Lambda}_e)^{-1}\mathbf{R}_e)\,\mathbf{w}(z) = \mathbf{0}. \tag{13}$$

This leads to a homogenous linear system on $\mathbf{w}(z)$ that allows one to identify a relation between $\mathbf{s}(z)$ and $\mathbf{s}_0(z)$. The latter may be partitioned into

$$[\mathbf{B}_0 \quad \mathbf{B}_s]\mathbf{w}(z) = \mathbf{0}, \tag{14}$$

where $\mathbf{B}_s$ has dimensions $(q \times n)$ and $\mathbf{B}_0$ has dimensions $(q \times q)$.

Combining Eqs. (4) and (14) and solving for $\mathbf{s}_0(z)$ leads to

$$\mathbf{s}_0(z) = -\mathbf{B}_0^{-1}\mathbf{B}_s\,\mathbf{s}(z). \tag{15}$$

Through Eq. (15), $\mathbf{w}(z)$ may be expressed as a function of the state vector $\mathbf{s}(z)$,

$$\mathbf{w}(z) = \mathbf{T}\,\mathbf{s}(z), \tag{16}$$

with

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{B}_0^{-1}\mathbf{B}_s \end{bmatrix}, \tag{17}$$

where $\mathbf{I}_n$ is the identity matrix of size n.

After insertion of Eq. (16) into Eq. (12), the partial derivative of $\boldsymbol{s}(z)$ with respect to z is obtained as

$$\frac{\partial}{\partial z}\mathbf{s}(z) = -(\mathbf{R}_e\mathbf{\Phi}_e\mathbf{\Lambda}_e\mathbf{\Psi}_e\mathbf{T})^{-1}\mathbf{R}_e\mathbf{T}\,\mathbf{s}(z). \tag{18}$$

Thus, by inspection, the matrix $\boldsymbol{\alpha}$ in Eq. (5) is identified and rewritten as

$$\boldsymbol{\alpha} = (\Psi_e\mathbf{A}_z\mathbf{T})^{-1}\Psi_e\mathbf{R}\mathbf{T}. \tag{19}$$

3. Conclusion

An alternative method to compute the state matrix of a linear homogeneous medium under a plane wave excitation is presented. The approach is based on an eigenvector and eigenvalue decomposition of a system of first order linear equations. By isolating the non-zero eigenvalues and eigenvectors, the state matrix is derived through a series of projections on the computed eigenvectors.

As a validation, the terms of the state matrix computed using the proposed method have been compared to the results in Parra Martinez *et al.*,[5] and found to be equal within the numerical precision.

Given the generality of the proposed method, any type of linear homogeneous medium, including arbitrary anisotropic properties and multiphysics interactions, may be treated.

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